

ON  $\mathbb{Q}$ -CONIC BUNDLES, II

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ABSTRACT. A  $\mathbb{Q}$ -conic bundle germ is a proper morphism from a threefold with only terminal singularities to the germ  $(Z \ni o)$  of a normal surface such that fibers are connected and the anti-canonical divisor is relatively ample. We obtain the complete classification of  $\mathbb{Q}$ -conic bundle germs when the base surface germ is singular. This is a generalization of [MP06], which further assumed that the fiber over  $o$  is irreducible.

## 1. INTRODUCTION

This note is a continuation of our previous work [MP06] where we studied the local structure of  $\mathbb{Q}$ -conic bundles.

**(1.1) Definition.** A  $\mathbb{Q}$ -conic bundle is a projective morphism  $f: X \rightarrow Z$  from a threefold with only terminal singularities to a surface such that

- (i)  $f_*\mathcal{O}_X = \mathcal{O}_Z$  and all fibers are one-dimensional,
- (ii)  $-K_X$  is  $f$ -ample.

For  $f: X \rightarrow Z$  as above and for a point  $o \in Z$ , we call the analytic germ  $(X, f^{-1}(o)_{\text{red}})$  a  $\mathbb{Q}$ -conic bundle germ.

In [MP06] we completely classified  $\mathbb{Q}$ -conic bundle germs over a singular base and such that the central fiber is irreducible. For convenience of quotations we reproduce briefly the classification. For more detailed explanations we refer to the original paper [MP06].

**(1.2) Theorem.** *Let  $f: (X, C) \rightarrow (Z, o)$  be a  $\mathbb{Q}$ -conic bundle germ, where  $C$  is irreducible and  $(Z, o)$  is singular. Then we are in one of the following cases:*

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Type	No.	singularities	$(Z, o)$
toroidal	(1.2.1)	$\frac{1}{n}(1, a, -a)$ and $\frac{1}{n}(-1, a, -a)$ , $\gcd(n, a) = 1$	$A_{n-1}$
(IA)+(IA)	(1.2.2)	$\frac{1}{n}(a, -1, 1)$ and $\frac{1}{n}(a+1, 1, -1)$ , $n = 2a + 1$	$A_{n-1}$
(IE $^\vee$ )	(1.2.3)	$\frac{1}{8}(5, 1, 3)$	$A_3$
(ID $^\vee$ )	(1.2.4)	$cA/2$ or $cAx/2$	$A_1$
(IA $^\vee$ )	(1.2.5)	$\frac{1}{4}(1, 1, 3)$ (+ (III))	$A_1$
(II $^\vee$ )	(1.2.6)	$cAx/4$ (+ (III))	$A_1$

In this paper we consider the case where the base surface is singular and the central fiber is reducible. Our main result is the following.

**(1.3) Theorem.** *Let  $f: (X, C) \rightarrow (Z, o)$  be a  $\mathbb{Q}$ -conic bundle germ. Assume that  $C$  is reducible and the base surface  $(Z, o)$  is singular. Then  $(Z, o)$  is Du Val of type  $A_1$  and  $(X, C)$  is the  $\mu_2$ -quotient of the index-two  $\mathbb{Q}$ -conic bundle  $f': (X', C') \rightarrow (Z', o')$  over a smooth base, where  $\mu_2$  acts on  $(Z', o')$  freely in codimension one. Moreover,  $C'$  has four irreducible components,  $\mu_2$  does not fix any of them and  $X$  has a unique non-Gorenstein point  $P$ . Furthermore,  $X'$  is given by the following two equations in  $\mathbb{P}(1, 1, 1, 2)_{y_1, \dots, y_4} \times \mathbb{C}_{u, v}^2$*

$$\begin{cases} y_1^2 - y_3^2 &= \psi_1(y_1, \dots, y_4; u, v), \\ y_2^2 - y_3^2 &= \psi_2(y_1, \dots, y_4; u, v), \end{cases}$$

where  $\mu_2$  acts as follows:

$$(y_1, y_2, y_3, y_4; u, v) \longmapsto (-y_1, -y_2, y_3, -y_4; -u, -v).$$

Here  $\psi_i = \psi_i(y_1, \dots, y_4; u, v)$  are weighted quadratic in  $y_1, \dots, y_4$  with respect to  $\text{wt}(y_1, \dots, y_4) = (1, 1, 1, 2)$  and  $\psi_i(y_1, \dots, y_4; 0, 0) = 0$ . The following are the only possibilities:

**(1.3.1)**  $(X, P)$  is a cyclic quotient singularity of type  $\frac{1}{4}(1, 1, -1)$  and for any component  $C_i \subset C$  germ  $(X, C_i)$  is of type (IA $^\vee$ ),

**(1.3.2)**  $(X, P)$  is a singularity of type  $cAx/4$  and for any component  $C_i \subset C$  germ  $(X, C_i)$  is of type (II $^\vee$ ).

Conversely, if the quotient  $(X, C) = (X', C')/\mu_2$ , where  $(X', C')$  and the action of  $\mu_2$  are as above, has only terminal singularities, then  $(X, C)$  is a conic bundle germ over  $\mathbb{C}_{u, v}^2/\mu_2$  with reducible central fiber  $C$ .

Below are a series of explicit examples of  $\mathbb{Q}$ -conic bundles as in (1.3).

**(1.3.3) Example.** Consider the subvariety  $X' \subset \mathbb{P}(1, 1, 1, 2) \times \mathbb{C}^2$  defined by the following two equations:

$$\begin{cases} y_1^2 - y_3^2 + u^{2k+1}y_4 + v^2y_2^2 &= 0, \\ y_2^2 - y_3^2 + vy_4 &= 0. \end{cases}$$

The projection  $f': X' \rightarrow \mathbb{C}^2$  is a  $\mathbb{Q}$ -conic bundle of index 2 (cf. [MP06, 12.1.3]). Define the action of  $\mu_2$  on  $X'$  as follows

$$(y_1, y_2, y_3, y_4; u, v) \longmapsto (-y_1, -y_2, y_3, -y_4; -u, -v).$$

Then  $X'/\mu_2 \rightarrow \mathbb{C}^2/\mu_2$  is a  $\mathbb{Q}$ -conic bundle with a unique non-Gorenstein point  $P$ . The point  $P$  is of type (1.3.1) if  $k = 0$  and of type (1.3.2) if  $k \geq 1$ .

The basic idea of the proof is to reduce the problem of classifying  $\mathbb{Q}$ -conic bundles  $(X, C)$  as in Theorem (1.3) to the case where the central fiber is irreducible by applying the MMP to a  $\mathbb{Q}$ -factorialization  $(X^q, C^q)$ . Then the resulting  $\mathbb{Q}$ -conic bundle  $(\bar{X}, \bar{C})$  belongs to the list (1.2). We trace back from  $(\bar{X}, \bar{C})$  to  $(X, C)$ . It turns out that in many cases the steps of the MMP do not affect the singularities of  $(\bar{X}, \bar{C})$ . Here we use some results about divisorial contractions and flips (see §2) based on [KM92] and [Kaw96]. Then the base change trick allows us to show that  $(X, C)$  is a  $\mu_2$ -quotient of an index-two conic bundle, see §3.

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## 2. PRELIMINARY RESULTS ON EXTREMAL CONTRACTIONS

**(2.1)** Let  $(E^\sharp, P^\sharp)$  be a Du Val singularity. (We assume that  $(E^\sharp, P^\sharp)$  is *singular*). Assume that  $\mu_m$  acts on  $E^\sharp$  freely outside  $P^\sharp$  and the quotient  $(E, P) = (E^\sharp, P^\sharp)/\mu_m$  is also Du Val. Then there is a  $\mu_m$ -equivariant embedding  $(E^\sharp, P^\sharp) \subset (\mathbb{C}_{x,y,z}^3, 0)$  such that  $x, y, z$  and the equation of  $E^\sharp$  are semi-invariant. Let  $F^\sharp \subset \mathbb{C}^3$  be the locus of points at which the action of  $\mu_m$  is not free. By our assumption  $F^\sharp$  is a curve. Define the invariant  $\varsigma(E^\sharp, P^\sharp, \mu_m)$  as the local intersection number

$(E^\sharp \cdot F^\sharp)_0$ . According to [Rei87, 4.10] we have only the following cases:

(2.1.1)

$m$	$(E^\sharp, P^\sharp) \rightarrow (E, P)$	$\varsigma(E^\sharp, P^\sharp, \boldsymbol{\mu}_m)$
any	$A_{r-1} \rightarrow A_{mr-1}$	$r$
4	$A_{2r-2} \rightarrow D_{2r+1}$	$2r - 1$
2	$A_{2r-1} \rightarrow D_{r+2}$	2
3	$D_4 \rightarrow E_6$	2
2	$D_{r+1} \rightarrow D_{2r}$	$r$
2	$E_6 \rightarrow E_7$	3

**(2.1.2)** Let  $(W, P)$  be a three-dimensional terminal singularity of index  $m > 1$  and let  $E \in |-K_W|$  be a divisor having a Du Val singularity at  $P$ . Assume that  $(W, P)$  is not a cyclic quotient. Let  $\pi: (W^\sharp, P^\sharp) \rightarrow (W, P)$  be the index-one  $\boldsymbol{\mu}_m$ -cover and let  $(W^\sharp, P^\sharp) = \{\phi = 0\} \subset \mathbb{C}_{x_1, x_2, x_3, x_4}^4$  be a  $\boldsymbol{\mu}_m$ -equivariant embedding. Let  $E^\sharp := \pi^{-1}(E)$  and  $F^\sharp \subset \mathbb{C}^3$  be the locus of points at which the action of  $\boldsymbol{\mu}_m$  is not free. Since  $\pi$  is free in codimension two,  $F^\sharp$  is a curve. Recall that the local intersection number  $(W^\sharp \cdot F^\sharp)_0$  is called the *axial multiplicity* of  $(W, P)$  [Mor88, 1a.5]. We denote it by  $\text{am}(W, P)$ . By the classification of terminal singularities we may assume that  $F^\sharp$  is the  $x_4$ -axis, and either  $\text{wt}(x_1, x_2, x_3, x_4, \phi) \equiv (1, -1, a, 0, 0) \pmod{m}$ , or  $m = 4$  and  $\text{wt}(x_1, x_2, x_3, x_4, \phi) \equiv (1, -1, a, 2, 2) \pmod{4}$ , where  $\gcd(a, m) = 1$ . Since  $(E^\sharp, P^\sharp)$  is a Du Val singularity, its Zariski tangent space at the origin is three-dimensional. Hence there is a  $\boldsymbol{\mu}_m$ -stable hypersurface  $H^\sharp \subset \mathbb{C}^4$  such that  $E^\sharp = H^\sharp \cap W^\sharp$  and  $H^\sharp$  is smooth.

**(2.1.3) Claim.**  $F^\sharp \subset H^\sharp$ .

*Proof.* Let  $\psi$  be the  $\boldsymbol{\mu}_m$ -semi-invariant equation of  $H^\sharp$ . Then  $\text{wt } \psi \equiv a$ . Hence  $\psi$  does not contain terms  $x_4^k$  and so it vanishes on  $F^\sharp$ .  $\square$

**(2.1.4)** We define the invariant  $\varsigma(W, E, P)$  as the local intersection number  $(E^\sharp \cdot F^\sharp)_0$  inside  $H^\sharp$ . Clearly it coincides with  $\varsigma(E^\sharp, P^\sharp, \boldsymbol{\mu}_m)$  defined above.

**(2.1.5) Lemma.** Assume that  $(W, P)$  is not a cyclic quotient singularity. The invariant  $\varsigma(W, E, P)$  does not depend on the choice of  $E$  and  $\varsigma(W, E, P) = \text{am}(W, P)$ .

*Proof.* Both sides of the equality coincide with the order of vanishing of  $\phi|_{F^\sharp}$ .  $\square$

**(2.1.6) Corollary.** *Let  $(W, P)$  is a three-dimensional terminal singularity of index  $m > 1$  which is not a cyclic quotient and let  $E \in |-K_{(W,P)}|$  be a member having a Du Val singularity of  $A$ -type at  $P$ . Then  $E$  is isomorphic to a general member  $E_{\text{gen}} \in |-K_{(W,P)}|$ .*

*Proof.* By the above lemma we have  $\varsigma(E^\sharp, P^\sharp, \boldsymbol{\mu}_m) = \varsigma(E_{\text{gen}}^\sharp, P^\sharp, \boldsymbol{\mu}_m) = \text{am}(W, P)$ . Then the statement follows by the first line in (2.1.1).  $\square$

**(2.2) Proposition.** *Let  $\varphi: (V, \Gamma) \rightarrow (W, o)$  be the analytic germ of a divisorial extremal contraction of threefolds with terminal singularities (in particular,  $W$  is  $\mathbb{Q}$ -Gorenstein) such that the central fiber  $\Gamma := \varphi^{-1}(o)_{\text{red}}$  is one-dimensional and irreducible.*

- (i) *The point  $(W, o)$  cannot be of type  $cAx/4$ .*
- (ii) *If  $(W, o)$  is of type  $cAx/2$ , then  $(V, \Gamma)$  has a unique non-Gorenstein point which is of type  $(\text{II}^\vee)$ .*
- (iii) *If  $(W, o)$  is analytically isomorphic to*

$$(2.2.1) \quad \{x_1x_2 + x_3^2 + x_4^{2k} = 0\}/\boldsymbol{\mu}_2(1, 1, 0, 1),$$

*then  $(V, \Gamma)$  has a unique non-Gorenstein point  $P$  which is locally imprimitive of index 4 and splitting degree 2. Moreover,  $P \in (V, \Gamma)$  is either of type  $(\text{II}^\vee)$  or  $(\text{IA}^\vee)$  and in the second case  $(X, P)$  is a cyclic quotient singularity.*

*Proof.* For the proof we assume that  $(W, o)$  is of type  $cAx/4$ ,  $cAx/2$ , or as in (2.2.1). We will use the classification [KM92, Th. 2.2]. Let  $m$  be the index of  $(W, o)$ . Then the canonical class  $K_W$  is an  $m$ -torsion element in  $\text{Cl}^{\text{sc}}(W, o)$ . Its pull-back  $\varphi^*K_W$  is a well-defined Cartier divisor on  $V \setminus \Gamma$  such that  $m(\varphi^*K_W) \sim 0$ . Hence the group  $\text{Cl}^{\text{sc}}(V, \Gamma)$  contains an  $m$ -torsion element, say  $\xi$ . By the classification [KM92, Th. 2.2]  $\text{Cl}^{\text{sc}}(V, \Gamma)$  can contain a torsion only when  $(V, \Gamma)$  is of type (k1A) (with a point of type  $(\text{IA}^\vee)$ ),  $(\text{II}^\vee)$ , or (k2A).

Assume that  $(V, \Gamma)$  is of type (k2A). Then by [KM92, Th. 2.2] a general member  $D \in |-K_V|$  and its image  $\varphi(D) \in |-K_W|$  have only Du Val singularities. Moreover,  $(\varphi(D), o)$  is a singularity of type  $A_*$  and so  $(W, o)$  is of type  $cA/*$ . Clearly, the contraction  $\varphi|_D: D \rightarrow \varphi(D)$  is crepant. By our assumptions  $(W, o)$  is a singularity given by (2.2.1). So,  $\text{am}(W, o) = 2$ . By Corollary (2.1.6) the singularity  $(\varphi(D), o)$  is of type  $A_3$ . Since  $\varphi_D: D \rightarrow \varphi(D)$  is crepant and  $V$  has two singular points, the only possibility is that  $D$  has two singularities of type  $A_1$ . But in this case  $V$  is of index two and then by [KM92, Th. 4.7]  $V$  has a unique non-Gorenstein point, a contradiction.

In the remaining cases  $(\text{II}^\vee)$  and (k1A),  $V$  has a unique non-Gorenstein point  $P$ . Then  $(V, \Gamma)$  is locally imprimitive at  $P$  and the

splitting degree equals  $m$ . In particular, the index of  $P$  is  $> m$  [Mor88, Cor. 1.16]. Thus if  $(V, \Gamma)$  is of type  $(\text{II}^\vee)$ , then we are in the case (ii) or (iii).

Assume that  $(V, \Gamma)$  is of type  $(\text{k1A})$ . Then by [KM92, Th. 2.2] a general member  $D \in |-K_V|$  does not contain  $\Gamma$ , has only Du Val singularity at  $P := \{D \cap \Gamma\}$ , and  $\varphi|_D: D \rightarrow \varphi(D)$  is an isomorphism. Hence  $\varphi(D) \in |-K_W|$  has a Du Val singularity of type  $A$  at  $o$ . In this case,  $(W, o)$  cannot be of type  $cAx/*$ . Thus  $(W, o)$  is given by (2.2.1). By Corollary (2.1.6)  $D \simeq \varphi(D)$  is of type  $A_3$ . Since the index of  $(V, P)$  is  $> 2$ ,  $(V, P)$  must be a cyclic quotient singularity  $\frac{1}{4}(1, 1, -1)$ . So we are in the case (iii). This proves the proposition.  $\square$

**(2.3) Proposition.** *Let  $\chi: (V, \Gamma) \dashrightarrow (V^+, \Gamma^+)$  be a flip of threefolds with terminal singularities with irreducible flipping curve  $\Gamma$ . Then  $(V^+, \Gamma^+)$  contains none of the following configurations of singularities:*

- (i) *two cyclic quotient singularities  $P_1^+$  and  $P_2^+$  of indices  $m_1$  and  $m_2$  with  $\gcd(m_1, m_2) > 1$  such that  $(V^+, \Gamma^+)$  is locally primitive at  $P_1^+$  and  $P_2^+$ ;*
- (ii) *an imprimitive point  $P^+$  of splitting degree  $s > 1$ .*

*Proof.* By [KM92, Cor. 13.4]  $\Gamma^+$  is irreducible. Assume that one of the cases (i)-(ii) holds. As in [Mor88, Cor. 1.12] there is a  $d$ -torsion element  $\xi^+ \in \text{Cl}^{\text{sc}} V^+$  for some  $d > 1$ . Its proper transform  $\xi$  on  $V$  is a  $d$ -torsion element in  $\text{Cl}^{\text{sc}} V$ . In [KM92] flips are classified into 6 types  $(\text{k1A})$ ,  $(\text{k2A})$ ,  $(\text{cD}/3)$ ,  $(\text{IIA})$ ,  $(\text{IC})$ ,  $(\text{kAD})$  according to a general member of the anticanonical linear system  $|-K_V|$  [KM92, Th. 2.2]. The group  $\text{Cl}^{\text{sc}} V$  can contain a torsion only in cases  $(\text{k1A})$  and  $(\text{k2A})$  (in all other cases the flipping variety is locally primitive and indices of non-Gorenstein points are coprime, cf. [Mor88, Cor. 1.12]). The torsion elements  $\xi$  and  $\xi^+$  induce the following cyclic  $\mu_d$ -coverings:

$$(2.3.1) \quad \begin{array}{ccc} (V', \Gamma') & \xrightarrow{\chi'} & (V^{+'}, \Gamma^{+'}) \\ \downarrow \pi & & \downarrow \pi^+ \\ (V, \Gamma) & \xrightarrow{\chi} & (V^+, \Gamma^+) \end{array}$$

Consider the flipping diagram

$$\begin{array}{ccc} (V, \Gamma) & \xrightarrow{\chi} & (V^+, \Gamma^+) \\ \searrow \varphi & & \swarrow \varphi^+ \\ & (W, o) & \end{array}$$

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By [Mor88, Th. 7.3, 9.10] and [KM92, Th. 2.2], a general member  $D \in |-K_V|$  has only Du Val singularities. Since the restriction  $\varphi_D: D \rightarrow \varphi(D)$  is crepant, the same holds for  $\varphi(D) \in |-K_W|$ . Further, if we put  $D^+ = \chi(D)$ , then  $D^+ \in |-K_{V^+}|$  and  $D^+$  also has only Du Val singularities. Since  $K_{V^+} \cdot \Gamma^+ > 0$ ,  $D^+ \supset \Gamma^+$ .

**(2.3.2)** First we consider the case where our flip is of type (k1A). Then  $V$  has a unique non-Gorenstein point  $P$  and  $P$  is of type  $cA/*$ . In this case  $D \cap \Gamma = \{P\}$  and  $(\varphi(D), o) \simeq (D, P)$  is of type  $A_*$ . Since  $\text{Cl}^{\text{sc}} V$  has a torsion,  $(V, \Gamma)$  is locally imprimitive at  $P$ .

**(2.3.3)** Assume that we are in the case (i). We claim that  $V^+$  has at least one Gorenstein singular point. Indeed, since the germ  $(V, \Gamma)$  has only one non-Gorenstein point, it is locally imprimitive and in the diagram (2.3.1)  $\pi$  is the splitting cover [Mor88, Cor. 1.12]. Here  $\Gamma'$  has exactly  $d$  components and  $V^{+'}$  is the relative canonical model of  $V'$ . Since  $(V^+, \Gamma^+)$  is locally primitive at  $P_1^+$  and  $P_2^+$ , the curve  $\Gamma^{+'}$  is irreducible. Now the map  $\chi'$  can be decomposed as follows

$$\chi': V' = V'_0 \dashrightarrow V'_1 \dashrightarrow \cdots \dashrightarrow V'_n \rightarrow V^{+'},$$

where every  $V'_i \dashrightarrow V'_{i+1}$  is a flip along an irreducible curve and  $V'_n \rightarrow V^{+'}$  is a crepant small contraction (cf. [KM92, Proof of 13.5]). Every step  $V'_i \dashrightarrow V'_{i+1}$  preserves the number of components of the central fiber. Hence the crepant contraction  $V'_n \rightarrow V^{+'}$  is nontrivial and gives us a Gorenstein non- $\mathbb{Q}$ -factorial point  $Q \in \Gamma^+ \subset V^+$ . This proves our claim. Thus the divisor  $D^+$  has at least three singular points:  $P_1^+$ ,  $P_2^+$ , and  $Q$ . But then  $\varphi_D^+: D^+ \rightarrow \varphi(D)$  contracts  $\Gamma^+$  to a Du Val singularity of type  $D_*$  or  $E_*$ , a contradiction.

**(2.3.4)** Now we assume that we are in the case (ii). We claim that the log divisor  $K_{D^+} + \Gamma^+$  is not plt at  $P^+$ . Indeed, in the diagram (2.3.1)  $\pi^+$  is the splitting cover (see [Mor88, Cor. 1.12.1]). In particular,  $\pi^+$  is étale outside  $P^+$ ,  $\pi^{+-1}(P^+)$  is one point, and  $\Gamma^{+'}$  has  $s > 1$  irreducible components, all of them pass through  $\pi^{+-1}(P^+)$ . Let  $D^{+'} := \pi^{+-1}(D^+)$ . Since  $\Gamma^{+'}$  is singular at  $\pi^{+-1}(P^+)$ , the log divisor  $K_{D^{+'}} + \Gamma^{+'}$  is not plt at this point. This proves our claim because the restriction  $\pi_D^+: D^{+'} \rightarrow D^+$  is étale in codimension one (see, e.g., [Kol92, Cor. 20.4]). Now since the contraction  $D^+ \rightarrow \varphi(D)$  is crepant,  $D^+$  is dominated by the minimal resolution  $D^{\min}$  of  $\varphi(D)$ :  $D^{\min} \rightarrow D^+ \rightarrow \varphi(D)$ . Since  $K_{D^+} + \Gamma^+$  is not plt, the exceptional divisor of  $D^{\min} \rightarrow \varphi(D)$  is not a chain of smooth rational curves. Hence  $(\varphi(D), o)$  is not a singularity of type  $A_*$ , a contradiction.

**(2.3.5)** Finally, we consider the case where our flip is of type (k2A). These flips are described in [Mor02]. We will use notation of [Mor02].

By [Mor02, Th. 4.7]  $(V^+, \Gamma^+)$  is locally primitive. Hence we have the case (i). Moreover,  $V^+$  has exactly two singular points and they are analytically isomorphic to germs of the following  $cA/m_i$  singularities:

$$\{\xi_i \eta_i = G_{k-i}(\zeta_i^{m_i}, u^{e(k+2-i)})\} / \mu_{m_i} \subset \mathbb{C}_{\xi_i, \eta_i, \zeta_i, u}^4 / \mu_{m_i}(1, -1, a_i, 0),$$

where  $k, a_i$  are some positive numbers and  $e(j)$  is some function. Hence these points coincide with  $P_1^+$  and  $P_2^+$ . Since  $P_i^+ \in \Gamma^+ \subset V^+$  are cyclic quotient singularities, we have  $e(k) = e(k+1) = 1$  ( $u$  needs to be eliminated). If we put  $\delta := a_1 m_2 + a_2 m_1 - m_1 m_2$ , then  $\delta \geq d$  and by definition [Mor02, Def. 3.2] we have  $e(3) = 0$ ,  $e(4) = \delta \alpha_1 \geq d > 1$ ,  $e(5) = (\delta^2 \rho_2 - 1) \alpha_1 + \delta \alpha_2 \geq d > 1$  (see [Mor02, Rem. 3.6]). Thus,  $k \geq 6$ . On the other hand, by [Mor02, Lemma 3.5, Cor. 3.7] we have  $k \leq 5$ , a contradiction.

□

**(2.4) Proposition.** *Let  $\varphi: (V, \Gamma) \rightarrow (W, o)$  be the germ of a birational crepant contraction of threefolds with terminal singularities, where  $\Gamma$  is irreducible.*

- (i)  *$(V, \Gamma)$  contains at most two non-Gorenstein points.*
- (ii) *If  $(V, \Gamma)$  is imprimitive at some point  $P$ , then  $(W, o)$  cannot be a singularity of type  $cA/*$ .*

*Proof.* For the proof we assume that  $V$  is not Gorenstein. Since  $\varphi$  is crepant, the point  $(W, o)$  is not Gorenstein. Let  $m$  be its index. Let  $D \in |-K_{(W, o)}|$  be a general member and let  $S := \varphi^{-1}(D)$ . Then  $S \in |-K_{(V, \Gamma)}|$  and both  $S$  and  $D$  have only Du Val singularities. Moreover, the restriction map  $\varphi_S: S \rightarrow D$  is crepant. Hence  $S$  is dominated by the minimal resolution  $D^{\min}$  of  $D$  and obtained from  $D^{\min}$  by contracting all but one exceptional curves.

First assume that  $(V, \Gamma)$  has at least three non-Gorenstein points, say  $P, Q$ , and  $R$ . By the classification of Du Val singularities  $(D, o)$  is a singularity of type  $D_*$  or  $E_*$  and  $S$  is obtained from  $D$  by blowing up the exceptional curve corresponding to the central vertex in the Dynkin diagram. In this case exceptional curves on  $D^{\min}$  over  $(S, P)$ ,  $(S, Q)$  and  $(S, R)$  form strings and the proper transform of  $\Gamma$  is adjacent to the ends of them. This means that the log divisor  $K_S + \Gamma$  is plt. The latter implies that the germ  $(V, \Gamma)$  is locally primitive (cf. (2.3.4)). Now consider the index-one cover  $\pi: (W^\sharp, o^\sharp) \rightarrow (W, o)$ . It



induces the following diagram

$$(2.4.1) \quad \begin{array}{ccc} (V^\sharp, \Gamma^\sharp) & \xrightarrow{v} & (V, \Gamma) \\ \downarrow \varphi^\sharp & & \downarrow \varphi \\ (W^\sharp, o^\sharp) & \xrightarrow{\pi} & (W, o) \end{array}$$

Since  $(V, \Gamma)$  is locally primitive,  $\Gamma^\sharp = \pi^\sharp(o^\sharp)$  is irreducible. The group  $\mu_m$  naturally acts on  $\Gamma^\sharp \simeq \mathbb{P}^1$  and has exactly two fixed points. Thus we may assume that  $v^{-1}(R)$  contains no fixed points. But then  $v^{-1}(R)$  consists of  $m > 1$  non-Gorenstein points of the same index. By [Mor88, Cor. 1.12] there is a torsion element in  $\text{Cl}^{\text{sc}}(V^\sharp, \Gamma^\sharp) \simeq \text{Cl}^{\text{sc}}(W^\sharp, o^\sharp)$ . This contradicts the fact that  $W^\sharp \setminus \{o^\sharp\}$  is simply connected. Thus (i) is proved.

Now assume that  $(V, \Gamma)$  contains an imprimitive point  $P$ . By the proof of (i)  $S$  has at most two singular points and the log divisor  $K_S + \Gamma$  is not plt at  $P$ . On the other hand, assume that  $(D, o)$  is a point of type  $A_*$ . Then the exceptional curves of the minimal resolution  $D^{\text{min}} \rightarrow S$  and  $\Gamma$  form a chain. Hence  $K_S + \Gamma$  is not plt, a contradiction.  $\square$

**(2.5) Proposition (cf. [Mor88, 1.14]).** *Let  $f: (X, C) \rightarrow (Z, o)$  be the germ of a contraction from a threefold with only terminal singularities to a surface such that*

- (i)  $-K_X$  is nef and big,
- (ii)  $C := f^{-1}(o)_{\text{red}}$  is a curve having at least three components,
- (iii) each  $K_X$ -trivial component  $C_j \subset C$  contains a non-Gorenstein point.

*Then  $X$  has index  $> 1$  at all singular points of  $C$ .*

*Proof.* By the Kawamata-Viehweg vanishing theorem we have  $R^1 f_* \mathcal{O}_X = 0$ . Hence  $C$  is a union of  $\mathbb{P}^1$ 's whose configuration is a tree. Let  $P \in C$  be a singular point and let  $C_i \subset C$  be a component passing through  $P$ . We have  $\text{gr}_{C_i}^0 \omega \simeq \mathcal{O}(-1)$ . Indeed, take a positive integer  $m$  such that  $mK_X$  is Cartier. Then there is a natural embedding  $(\text{gr}_{C_i}^0 \omega)^{\otimes m} \hookrightarrow \mathcal{O}_{C_i}(mK_X)$ . Since  $K_X \cdot C_i \leq 0$  we have  $\deg \text{gr}_{C_i}^0 \omega \leq 0$ . Moreover, if  $K_X \cdot C_i < 0$ , then  $\deg \text{gr}_{C_i}^0 \omega < 0$ . Assume that  $K_X \cdot C_i = 0$ . Since  $C_i$  contains a non-Gorenstein point, the above embedding is not an isomorphism and so again  $\deg \text{gr}_{C_i}^0 \omega < 0$ . On the other hand,  $C_i$  is contractible over  $Z$ . Hence, by the Grauert-Riemenschneider vanishing theorem we have  $H^1(\text{gr}_{C_i}^0 \omega) = 0$ . This shows  $\text{gr}_{C_i}^0 \omega \simeq \mathcal{O}(-1)$ .

Now let  $C_j$  be another component of  $C$  passing through  $P$ . As above,  $\text{gr}_{C_j}^0 \omega \simeq \mathcal{O}(-1)$ . Consider the following exact sequence

$$0 \longrightarrow \text{gr}_{C_i \cup C_j}^0 \omega \longrightarrow \text{gr}_{C_i}^0 \omega \oplus \text{gr}_{C_j}^0 \omega \longrightarrow \mathcal{F} \longrightarrow 0,$$

where  $\text{Supp } \mathcal{F} = P$ . Since  $C_i \cup C_j \neq C$ ,  $C_i \cup C_j$  is contractible over  $Z$  and again by the Grauert-Riemenschneider vanishing  $H^1(\text{gr}_{C_i \cup C_j}^0 \omega) = 0$ . This implies  $\text{gr}_{C_i \cup C_j}^0 \omega \simeq \text{gr}_{C_i}^0 \omega \oplus \text{gr}_{C_j}^0 \omega$ . So  $\text{gr}_{C_i \cup C_j}^0 \omega$  is not locally free at  $P$  and this point cannot be Gorenstein.  $\square$

### 3. THE PROOF OF THE MAIN THEOREM

In this section we prove Theorem (1.3).

**(3.1) Notation.** Let  $f: (X, C) \rightarrow (Z, o)$  be a  $\mathbb{Q}$ -conic bundle germ with reducible central fiber  $C$ . Then  $\rho(X/Z) > 1$ . Recall that according to [MP06, Th. 1.2.7]  $(Z, o)$  is either smooth or Du Val of type  $A$ . We assume that  $(Z, o)$  is singular of type  $A_{n-1}$ ,  $n \geq 2$ .

**(3.1.1) Lemma.** *Notation as above.*

- (i) *If  $(X, C)$  has a point  $P$  such that either*
  - (a)  *$P$  is of type  $cAx/4$ , or*
  - (b) *for each component  $C_i \subset C$  passing through  $P$  the germ  $(X, C_i)$  is locally imprimitive at  $P$ .*

*Then  $P$  is the only non-Gorenstein point on  $X$ .*

- (ii) *Conversely, if  $P$  is a unique non-Gorenstein point on  $X$ , then all the components  $C_i \subset C$  pass through  $P$  and the germ  $(X, C_i)$  is locally imprimitive at  $P$ . If furthermore  $(X, P)$  is of index 4, then  $(X, C)$  is a quotient of an index two  $\mathbb{Q}$ -conic bundle germ  $(X', C')$  over a smooth base by  $\mu_2$ , where the action is free in codimension one,  $C'$  has four irreducible components and  $\mu_2$  does not fix any of them.*

*Proof.* Let  $P \in X$  be a point as in (i). For each component  $C_i \subset C$  passing through  $P$  the germ  $(X, C_i)$  is an extremal neighborhood and by [KM92, Th. 2.2]  $(X, C_i)$  has no non-Gorenstein point other than  $P$ . Since each singular point of  $C$  is not Gorenstein [Kol99, Prop. 4.2], [MP06, 4.4.2] and  $C$  is connected,  $P$  is the only non-Gorenstein point on the whole  $X$ .

Now assume that  $P$  is the only non-Gorenstein point. Consider the base change [MP06, 2.4]:  $(X', C') \rightarrow (X, C)$ . Here  $(X', C')$  is a conic bundle germ over a smooth base and  $X' \rightarrow X$  is an étale outside  $P$   $\mu_n$ -cover. Thus  $(X, C) = (X', C')/\mu_n$ . If  $\mu_n$  fixes a component  $C'_i \subset C'$ , then there are two  $\mu_n$ -fixed points on  $C_i$  and they give us two non-Gorenstein points on  $X$ , a contradiction. So the first assertion of (ii) is proved.

Finally assume that  $(X, P)$  is of index 4. Since the index of  $(X, P)$  is divisible by  $n$ ,  $n = 4$  or  $2$ . If  $n = 4$ , then  $X'$  is Gorenstein. In this case, by [Pro97, Th. 2.4]  $C$  is irreducible, a contradiction. Thus

$n = 2$  and  $(X', C')$  is of index 2. By the above,  $\mu_2$  does not fix any component of  $C'$ . On the other hand,  $C'$  has at most four components [MP06, Th. 12.1]. Hence  $C'$  has exactly four components. This proves the lemma.  $\square$

**(3.1.2)** Let  $q: X^q \rightarrow X$  be a  $\mathbb{Q}$ -factorialization. (It is possible that  $q$  is the identity map.) Run the MMP over  $Z$ :  $X^q = X_0 \dashrightarrow X_{N+1} = \bar{X}$ . Since  $X/Z$  is a rational curve fibration,  $X_{N+1}$  is not a minimal model over  $Z$ . Therefore, at the end we get an extremal contraction  $\bar{f}: \bar{X} \rightarrow \bar{Z}$  of Fano type over  $Z$ . Since the composition  $f^q: X^q \rightarrow Z$  has only one-dimensional fibers,  $Z = \bar{Z}$  and  $X^q \dashrightarrow \bar{X}$  is a sequence of flips and extremal divisorial contractions that contract a divisor to a curve which is not contained in the fiber over  $o \in Z$ . Thus we have the following diagram:

$$\begin{array}{ccccccc}
 (X^q, C^q) & \xrightarrow{g_0} & (X_1, C_1) & \dashrightarrow & \cdots & \dashrightarrow & (X_N, C_N) \xrightarrow{g_N} (\bar{X}, \bar{C}) \\
 \downarrow q & \searrow f^q & & & & & \nearrow \bar{f} \\
 (X, C) & & & & & & \\
 & \searrow f & & & & & \\
 & & (Z, o) & & & & 
 \end{array}$$

Here each  $X_k$  has a morphism  $f_k: X_k \rightarrow Z$  with connected one-dimensional fibers and  $C_k := f_k^{-1}(o)$  is the central fiber (with reduced structure). Since  $\rho(\bar{X}/Z) = 1$ ,  $\bar{f}: \bar{X} \rightarrow \bar{Z}$  is a  $\mathbb{Q}$ -conic bundle with irreducible central fiber  $\bar{C}$ . Since the base  $(Z, o)$  is singular,  $\bar{X}$  is not Gorenstein. So  $\bar{f}$  is classified in [MP06], see also (1.2).

**(3.1.3)** Note that each component of the central fiber  $C_k$  is contractible and the resulting variety is again projective over  $Z$  (because it has one-dimensional fibers over  $Z$ ). Hence each component of  $C_k$  generates an extremal ray (not necessarily  $K$ -negative). This implies that all our flipping curves are irreducible and all the divisorial contractions have irreducible fibers. Note also that all the varieties  $X_k$  are analytically  $\mathbb{Q}$ -factorial at each point on  $C_k$  (again because  $X_k \rightarrow Z$  has one-dimensional fibers, cf. [Mor88, Proof of 1.7]).

The following is the key argument in the proof.

**(3.2) Proposition.** *In the above notation one of the following holds.*

**(3.2.1)** *There is a component  $C_0^q \subset C^q$  containing two cyclic quotient singularities  $P^q$  and  $Q^q$  of index  $n$ . No other components of  $C^q$  pass through  $P^q$  and  $Q^q$ .*

**(3.2.2)** *There is a point  $P^q \in (X^q, C^q)$  of index  $m > 1$  which is contained in only one component  $C_0^q \subset C^q$  and such that  $(X^q, C_0^q)$  is locally imprimitive at  $P^q$ . The following are the possibilities for  $(n, m)$ :  $(4, 8)$ ,  $(2, 4)$ , and  $(2, 2)$ .*

**(3.2.3)** *There is a point  $P^q \in (X^q, C^q)$  which is contained in exactly two components  $C_0^q, C_1^q \subset C^q$  and such that both germs  $(X^q, C_i^q)$  are locally imprimitive at  $P^q$ . The point  $(X^q, P^q)$  is of type  $cAx/4$  or  $\frac{1}{4}(1, 1, -1)$ . Here  $n = 2$ .*

*Moreover, there is an  $n$ -torsion element  $\xi^q \in \text{Cl}^{\text{sc}}(X^q, C^q)$  which is not Cartier at  $P^q$  (and at  $Q^q$  is the case (3.2.1)).*

*Proof.* Since  $(Z, o)$  is of type  $A_{n-1}$ , there is an  $n$ -torsion element  $\eta \in \text{Cl}(Z, o)$ . Put  $\bar{\xi} := \bar{f}^*\eta$ ,  $\xi_l := f_l^*\eta$ , and  $\xi^q := f^{q*}\eta$ .

Assume that  $(\bar{X}, \bar{C})$  is either toroidal or of type (IA)+(IA). Let  $\bar{P}, \bar{Q}$  be the singular points of  $\bar{X}$ . Then  $\bar{\xi}$  is not Cartier at  $\bar{P}$  and  $\bar{Q}$ . We claim that the map  $\psi: \bar{X} \dashrightarrow X^q$  is an isomorphism near  $\bar{P}$  and  $\bar{Q}$ . Indeed, by induction, since  $\bar{P}, \bar{Q}$  are cyclic quotient singularities of index  $n$ , there is no divisorial contractions over these points by [Kaw96] and by Proposition (2.3) on each step the proper transform of  $\bar{C}$  cannot be a flipped curve. So if we put  $P^q := \psi(\bar{P})$ ,  $Q^q := \psi(\bar{Q})$ , and  $C_0^q := \psi(\bar{C})$ , we get the case (3.2.1).

Now assume that  $(\bar{X}, \bar{C})$  is of type  $(\text{IE}^\vee)$ ,  $(\text{IA}^\vee)$ , or  $(\text{II}^\vee)$ . Let  $\bar{P}$  be a (unique) non-Gorenstein point. Then  $(\bar{X}, \bar{P})$  is either a cyclic quotient singularity or of type  $cAx/4$  and again  $\bar{\xi}$  is not Cartier at  $\bar{P}$ . Moreover,  $(\bar{X}, \bar{C})$  is locally imprimitive at  $\bar{P}$ . As above, there is no divisorial contractions over  $\bar{P}$  by [Kaw96] and Proposition (2.2) and the proper transform of  $\bar{C}$  cannot be a flipped curve by Proposition (2.3). Put  $P^q := \psi(\bar{P})$  and  $C_0^q := \psi(\bar{C})$ . We get the case (3.2.2).

Finally consider the case where  $(\bar{X}, \bar{C})$  is of type  $(\text{ID}^\vee)$ . Then  $n = 2$ , i.e.,  $(Z, o)$  is of type  $A_1$ . Let  $\bar{P}$  be a (unique) non-Gorenstein point. Then  $(\bar{X}, \bar{C})$  is locally imprimitive at  $\bar{P}$  and  $(\bar{X}, \bar{P})$  is of type  $cA/2$  or  $cAx/2$ . Moreover, in the first case,  $(\bar{X}, \bar{P})$  is analytically isomorphic to a singularity given by (2.2.1). If there is no divisorial contractions over  $\bar{P}$ , we can argue as above and get the case (3.2.2). Otherwise on some step, the map  $\psi_{k+1}: \bar{X} \dashrightarrow X_{k+1}$  is an isomorphism near  $\bar{P}$  and there is a divisorial contraction  $g_k: X_k \rightarrow X_{k+1}$  which blows up a curve passing through  $P_{k+1} := \psi_{k+1}(\bar{P})$ . Let  $C_{k,0} := g_k^{-1}(P_{k+1})$  and let  $C_{k,1}$  be the proper transform of  $\bar{C}$  on  $X_k$ . By Proposition (2.2)  $X_k$  has exactly one non-Gorenstein point  $P_k$  on  $C_{k,0}$ . Moreover,  $P_k$  is either a cyclic quotient singularity  $\frac{1}{4}(1, 1, -1)$  or of type  $cAx/4$  and  $(X_k, C_{k,0})$  is locally imprimitive at  $P_k$  of splitting degree 2. Note that  $\xi_k = g_k^*\xi_{k+1}$  is non-Cartier at all points of  $C_{k,0}$ . Since  $P_k$  is the only

non-Gorenstein point on  $C_{k,0}$ ,  $\xi_k$  is not Cartier at  $P_k$ . Now if  $C_{k,1}$  does not pass through  $P_k$ , then as above we get the case (3.2.2). Assume that  $C_{k,0} \cap C_{k,1} = \{P_k\}$ .

We claim that  $(X_k, C_{k,1})$  is locally imprimitive at  $P_k$ . Indeed,  $\xi_k$  defines the double cover  $\pi_k: (X'_k, C'_k) \rightarrow (X_k, C_k)$  which is étale outside  $\text{Sing } X_k$ . Since  $\xi_k$  is not Cartier at  $P_k$ ,  $\pi_k$  does not split over  $P_k$ . Hence,  $C'_{k,1} := \pi_k^{-1}(C_{k,1})$  is connected. On the other hand, since  $(\bar{X}, \bar{C})$  is locally imprimitive at  $\bar{P}$ , the curve  $C'_{k,1}$  is reducible. This means that  $C_{k,1}$  is locally imprimitive at  $P_k$ . Finally as above the map  $X_k \dashrightarrow X^q$  is an isomorphism near  $P_k$ . We get case (3.2.3).  $\square$

**(3.3) Proposition.** *Notation as in (3.1). Then  $(X, C)$  contains only one non-Gorenstein point  $P$ . This point is either a cyclic quotient  $\frac{1}{4}(1, 1, -1)$  or of type  $cAx/4$ . Moreover, for each component  $C_i \subset C$  the germ  $(X, C_i)$  is imprimitive at  $P$  and  $(Z, o)$  is of type  $A_1$ .*

*Proof.* By Proposition (3.2) there is a component  $C_0^q \subsetneq C^q$  as in (3.2.1), (3.2.2), or (3.2.3). First assume that  $C_0^q$  is not contracted by  $q: X^q \rightarrow X$ . Put  $C_0 := q(C_0^q)$ . Then  $(X, C_0)$  is an extremal neighborhood. In the case (3.2.1) it has two cyclic quotient singularities at  $q(P^q)$  and  $q(Q^q)$  and no other components of  $C$  pass through  $q(P^q)$  and  $q(Q^q)$ . On the other hand,  $C \neq C_0$  and intersection points  $C_0 \cap (C - C_0)$  are non-Gorenstein [Kol99, Prop. 4.2], [MP06, 4.4.2]. Thus the extremal neighborhood  $(X, C_0)$  has at least three non-Gorenstein points. This contradicts [Mor88, Th. 6.2]. Similarly, in the case (3.2.2),  $(X, C_0)$  is locally imprimitive at  $q(P^q)$  and no other components of  $C$  pass through  $q(P^q)$ . We get a contradiction by Lemma (3.1.1). Consider the case (3.2.3). If  $C_1^q$  is not contracted by  $q$ , then we are done by Lemma (3.1.1). If  $C_1^q$  is contracted by  $q$ , then  $q(C_1)$  is a point of type  $cAx/4$  by Proposition (2.4) and because  $P^q$  is of index 4. Then again the assertion follows by Lemma (3.1.1).

From now on we assume that  $q$  contracts  $C_0^q$ , i.e.,  $K_{X^q} \cdot C_0^q = 0$ . In the case (3.2.3) by symmetry and by the above arguments we may assume that  $q$  contracts  $C_1^q$ . Consider the decomposition

$$q: X^q \xrightarrow{\varphi} X^\delta \xrightarrow{\delta} X,$$

where  $\varphi$  contracts all the  $K_{X^q}$ -trivial components of  $C^q$  except for  $C_0^q$ . Put  $C^\delta := \varphi(C^q)$  and  $C_0^\delta := \varphi(C_0^q)$ . Thus  $-K_{X^\delta}$  is nef and big over  $Z$  and  $C_0^\delta$  is the only  $K_{X^\delta}$ -trivial curve on  $X^\delta/Z$ . Let  $C^{\delta\delta} := C^\delta - C_0^\delta$ . Then  $C^{\delta\delta}$  has at least two components. Let  $P := \delta(C_0^\delta)$  and  $R^\delta = C^{\delta\delta} \cap C_0^\delta$ . By Proposition (2.5)  $R^\delta$  is not Gorenstein.

In the case (3.2.1),  $C_0^\delta$  contains at least three non-Gorenstein points:  $R^\delta$ ,  $P^\delta := \varphi(P^q)$ , and  $Q^\delta := \varphi(Q^q)$ . This contradicts Proposition (2.4).

In the case (3.2.2),  $P^\delta := \varphi(P^q)$  is a locally imprimitive point of  $(X^\delta, C_0^\delta)$ . By Proposition (2.4) the singularity  $(X, P = \delta(C_0^\delta))$  is not of type  $cA/*$ . If the index of  $(X, P)$  is  $\geq 4$ , then  $(X, P)$  is of type  $cAx/4$  and we can apply Lemma (3.1.1). Thus we assume that  $(X, P)$  is of index 2 and  $n = 2$ . Let  $C_i \subset C$  be a component passing through  $P$ . By [Mor88, Cor. 1.16]  $(X, C_i)$  is primitive at  $P$ . Further,  $\xi := f^*\eta = q_*\xi^q$  is an 2-torsion element of  $\text{Cl}^{\text{sc}}(X, C)$  and is not Cartier at  $P$ . This defines a double étale in codimension one cover  $(X', C'_i) \rightarrow (X, C_i)$  which does not split over  $P$ . Hence there is a point  $Q \in (X, C_i)$  of even index. This contradicts the classification [KM92, Th. 2.2] (cf. [Mor07]).

Consider the case (3.2.3). Then  $P^\delta := \varphi(P^q)$  is a point of index  $\geq 4$  (because  $\varphi$  is a crepant contraction). Recall that  $\varphi$  contracts  $C_1^q$  by our assumption. Then by Proposition (2.4)  $(X^\delta, P^\delta)$  is a point of type  $cAx/4$ . As in the proof of Proposition (2.4), let  $D \in |-K_{(X, \delta(P^\delta))}|$  be a general element and let  $S := \delta^{-1}(D)$ . Then both  $D$  and  $S$  have only Du Val singularities and the contraction  $\delta_S: S \rightarrow D$  is crepant. Since  $(S, P^\delta)$  is not of type  $A_*$ , the germ  $(D, P)$  also cannot be of type  $A_*$ . Hence,  $(X, P)$  is not of type  $cA/*$  and so it is of type  $cAx/4$  (because its index is  $\geq 4$ ). Then the assertion follows by Lemma (3.1.1).  $\square$

**(3.4) Explicit forms.** By Proposition (3.3) and Lemma (3.1.1)  $f: (X, C) \rightarrow (Z, o)$  is a quotient of an index-two  $\mathbb{Q}$ -conic bundle  $f': (X', C') \rightarrow (Z', o')$  over a smooth base by  $\mu_2$ , where  $\mu_2$  acts on  $X'$  and  $Z'$  freely in codimension one. By [MP06, Prop. 12.1.10] there is a  $\mu_2$ -equivariant diagram

$$\begin{array}{ccc} X' & \hookrightarrow & \mathbb{P}(1, 1, 1, 2) \times \mathbb{C}^2 \\ & \searrow f & \downarrow p \\ & & \mathbb{C}^2 \end{array}$$

where the actions of  $\mu_2$  on  $(\mathbb{C}^2, 0) \simeq (Z', o')$  and  $\mathbb{P}(1, 1, 1, 2)$  are linear. Further, we can make coordinates  $y_1, y_2, y_3, u, v$  in  $\mathbb{P}(1, 1, 1, 2)$  and  $\mathbb{C}^2$  to be semi-invariant. By [MP06, Th. 12.1]  $X'$  is given by two semi-invariant equations

$$\begin{cases} q_1(y_1, y_2, y_3) - \psi_1(y_1, \dots, y_4; u, v) = 0, \\ q_2(y_1, y_2, y_3) - \psi_2(y_1, \dots, y_4; u, v) = 0, \end{cases}$$

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where  $\psi_i$  and  $q_i$  are weighted quadratic in  $y_1, \dots, y_4$  with respect to  $\text{wt}(y_1, \dots, y_4) = (1, 1, 1, 2)$  and  $\psi_i(y_1, \dots, y_4; 0, 0) = 0$ . Since the action of  $\mu_2$  on  $Z \simeq \mathbb{C}^2$  is free outside 0, this action is given by  $u \mapsto -u$ ,  $v \mapsto -v$ . Modulo multiplication on  $\pm 1$  and permutations of  $y_1, y_2, y_3$ , we may assume also that  $y_1 \mapsto -y_1$ ,  $y_2 \mapsto -y_2$ ,  $y_3 \mapsto y_3$ . Otherwise all the points of  $\{y_4 = 0\} \cap C'$  are fixed by  $\mu_2$ , while  $P$  is the only non-Gorenstein on  $X$ .

The central fiber  $C'$  is defined by  $q_1 = q_2 = 0$ . By Lemma (3.1.1)  $C'$  has exactly four components and  $\mu_2$  does not fix any of them. Thus we may assume that  $C' = \cup C'_i$ ,  $i = 1, 2, 3, 4$  and  $\mu_2$  interchanges  $C'_1$  and  $C'_2$  (resp.  $C'_3$  and  $C'_4$ ). For any two components  $C'_i \neq C'_j$  of  $C'$ , there is a linear form  $l_{i,j}(y_1, \dots, y_3)$  that vanishes along  $C'_i \cup C'_j$ . Then quadratic forms  $l_{1,2}l_{3,4}$ ,  $l_{1,3}l_{2,4}$ ,  $l_{1,4}l_{2,3}$  vanish along  $C'$ . Hence they belong to the pencil  $\lambda_1 q_1 + \lambda_2 q_2$  and semi-invariant. This implies that the action of  $\mu_2$  on the pencil is trivial. Moreover, we can put  $q_1 = l_{1,3}l_{2,4}$  and  $q_2 = l_{1,4}l_{2,3}$ . In view of the  $\mu_2$ -action we may assume that  $l_{1,3} = y_1 + y_3$ ,  $l_{2,4} = y_1 - y_3$ ,  $l_{1,4} = y_2 + y_3$ ,  $l_{2,3} = y_2 - y_3$  after some linear coordinate change of  $y_1, y_2, y_3$ .

We claim that  $y_4 \mapsto -y_4$ . The arguments below are similar to ones in the proof of [MP06, Lemma 12.1.12]. Assume to the contrary that  $y_4 \mapsto y_4$ . Let  $U \subset \mathbb{P}(1, 1, 1, 2)$  be the chart  $y_4 \neq 0$ . Then  $U \simeq \mathbb{C}_{z_1, z_2, z_3}^3 / \mu_2(1, 1, 1)$ . Let  $X^\sharp$  be the pull-back of  $X \cap (U \times \mathbb{C}_{u, v}^2)$  on  $\mathbb{C}_{z_1, z_2, z_3}^3 \times \mathbb{C}_{u, v}^2$  and let  $P^\sharp \in X^\sharp$  be the preimage of  $P$ . Since the induced map  $X^\sharp \rightarrow X$  is étale in codimension one,  $(X^\sharp, P^\sharp) \rightarrow (X, P)$  is the index-one cover. Hence  $(X^\sharp, P^\sharp) \rightarrow (X, P) / \mu_2$  is also the index-one cover of the terminal point  $(X, P) / \mu_2$  of index 4 (the last is true because the action of  $\mu_2$  is free in codimension one). Hence the morphism is a  $\mu_4$ -covering by the structure of terminal singularities. However  $(X, P) / \mu_2$  is the quotient of  $(X^\sharp, P^\sharp)$  by commuting  $\mu_2$ -actions:

$$(z_1, z_2, z_3, u, v) \mapsto (-z_1, -z_2, -z_3, u, v), (z_1, -z_2, z_3, -u, -v)$$

This is a contradiction, and we have  $y_4 \mapsto -y_4$  as claimed. This finishes the proof of Theorem (1.3).

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